Adiabatic radial pulsation theory

with some morsels of stellar structure theory

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Preludium: Linear pulsation theory deals with the oscillation frequencies and the growth properties of eigenmodes of small perturbations applied to a stellar equilibrium configuration. Before dealing with the calculus and main results of adiabatic pulsation theory, basic aspects of theory of stellar structure are quickly reiterated.

The framework: Rewriting the hydrodynamical conservation equations of a self-gravitating system with spherical symmetry leads eventually to the canonical equations for stellar structure and evolution.

> Marginalia: Spherical symmetry is not only a good idea, but it is the law; at least in Newton's flat universe and for a star devoid of rotation and any non-radial forces acting on it. As discussed in Lichtenstein (1918), Lichtenstein (1933), Carleman (1919), or Wavre (1932), isolated, nonrotating, self-gravitating gas spheres admit spherical symmetry. The generalization to General Relativity turned out to be surprisingly difficult. Significant progress with reasonable assumptions concerning the material functions was seen only in Massod-ul-Alam (1988) and Beig & Simon (1991). It is said, however, that significant steps in the proof are still to be taken.
> In rotating spheres, at least a plane of symmetry remains (this is of course the equatorial plane, when looking at the problem intuitively).

Hence, in the following we can rely on spherical symmetry not only for reasons of analytical simplicity but by generosity of nature. In spherical symmetry, with vectors expressed in spherical polar coordinates (r, θ, ϕ) we write, for example for the velocity

$$\mathbf{u} = (u_r, u_\theta, u_\phi) \equiv (u(r,t), 0, 0).$$

Detailed nonradial dynamics (such as nonradial waves, convection with its horizontal velocity fields, meridional circulation) must be treated in a *mean field* approach, i.e. by suitably averaging over spherical shells.

$$\overline{q} = \frac{\int q(r,\theta,\phi)\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}{\int \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}\,.$$

We decompose the energy flux as follows

$$\mathbf{F} = (F_r, 0, 0) = (F_\mathrm{R} + F_\mathrm{C} + F_\mathrm{cond}, 0, 0) \tag{1}$$

with F_R being the diffusive radiative flux, F_C the convective flux, and F_cond the thermal conductive flux due to Fermi-degenerate gas.

The macroscopic fluid equations in spherical symmetry (neglecting viscosity) read as:

basic fluid equations in spherical symmetry

$$\partial_t \rho = -\frac{1}{r^2}\partial_r\left(r^2 \rho u\right), \tag{2}$$

$$\mathrm{D}_t u = -\frac{1}{\rho}\,\partial_r P + g, \tag{3}$$

$$T\mathrm{D}_t s = -\frac{1}{\rho r^2}\,\partial_r\left(r^2 F_r\right) + \varepsilon, \tag{4}$$

$$F_r = F_\mathrm{R} + F_\mathrm{cond} + F_\mathrm{C}, \tag{5}$$

$$F_\mathrm{R} = -\frac{c}{3\kappa_\mathrm{R}\rho}\,\partial_r\left(aT^4\right), \tag{6}$$

$$F_\mathrm{cond} = -\mathcal{K}_\mathrm{c}\,\partial_r T, \tag{7}$$

$$\frac{1}{r^2}\,\partial_r\left(r^2 g\right) = -4\pi G\rho. \tag{8}$$

For the convective flux, F_C, no simple and commonly agreed on model and therefore no simple formula for F_C exists; an elementary exposition of convection is given in a later chapter. The quantity $\mathcal{K}_\mathrm{c}$ denotes the 'degenerate' thermal conductivity and we furthermore replaced $\nabla\Phi$ by the gravitational acceleration g. The total pressure is the sum $P = P_\mathrm{g} + P_\mathrm{rad}$ and we took advantage of the comoving time derivative

$$\mathrm{D}_t\,(.) = \partial_t\,(.) + u\cdot\partial_r\,(.). \tag{9}$$

For clarity, stellar-structure work refers mostly to an effective opacity κ that combines the effect of radiative diffusion and degeneracy

$$\frac{1}{\kappa} \doteq \frac{1}{\kappa_\mathrm{R}} + \left(\frac{3\rho}{4acT^3}\right)\mathcal{K}_\mathrm{c},$$

so that we can combine

$$F_\mathrm{R} + F_\mathrm{cond} = -\frac{c}{3\kappa\rho}\,\partial_r\left(aT^4\right).$$

Marginalia: For stellar evolution computations, the Lagrangian point of view is the most convenient one. During its evolution, a star experiences huge radius variations, so that measuring physical quantities in the mass scale (following, in conventional language, a particular mass shell) is computationally most favorable.

Going Lagrangian: Be $m(r,t)$ the mass enclosed in a sphere with radius r at time t

$$m \equiv \int \rho(r',t)\, 4\pi r'^2 \mathrm{d}r',$$

or differentially

$$\partial_r m = 4\pi\rho r^2. \tag{10}$$

Poisson's equation (8) can be integrated once to yield

$$r^2 g = -Gm; \tag{11}$$

this first integral of the Poisson equation can be performed easily only in spherical symmetry.

The continuity equation reduces to

$$\mathrm{D}_t m = 0$$

in the absence of processes that create or destroy matter.

Continuing to transform from the (r,t) basis to the (m,t) one using eq.(9) and

$$\begin{aligned}
\mathrm{D}_t\, r =& \partial_t\, r|_r + u \cdot \partial_r\, r|_t = 0 + u \\
\partial_r\, q|_t =& \partial_m r|_t \cdot \partial_m q|_t = 4\pi r^2 \rho\, \partial_m q|_t
\end{aligned}$$

for any physical quantity q, we can formulate the canonical stellar evolution equations:

stellar structure equations

$$\begin{aligned}
\partial_m r =& \frac{1}{4\pi r^2 \rho} & (12)\\
\mathrm{D}_t^2 r =& -4\pi r^2 \partial_m P - \frac{Gm}{r^2} & (13)\\
T\,\mathrm{D}_t s =& -\partial_m L + \varepsilon & (14)\\
L =& 4\pi r^2 (F_\mathrm{R} + F_\mathrm{cond} + F_\mathrm{C}) & (15)\\
=& \frac{64\pi^2 r^4 caT^3}{3\kappa} \partial_m T + 4\pi r^2 F_\mathrm{C}
\end{aligned}$$

As mentioned before, we need to adopt some convection modeling to compute the convective flux F_C.

Material functions need to be prescribed to completely define the problem:

Equation of state (EoS): Specification of EoS

$$s = f(P, T, \boldsymbol{\chi})$$
$$\rho = f(P, T, \boldsymbol{\chi})$$

where we have chosen $(P, T, \boldsymbol{\chi})$ to define the thermodynamic basis of the system. The nuclear composition of the star is expressed as $\boldsymbol{\chi}(m) = (\mathrm{X}(m), \mathrm{Y}(m), \mathrm{C}(m), ...)$. As mentioned earlier, the pressure is usually the sum of $P_{\mathrm{g}} + P_{\mathrm{rad}}$. Expressing $P_{\mathrm{rad}} \equiv aT^4/4$ is equivalent to assuming that radiation and matter are in equilibrium. Evolutionary or structural changes in stars happen on time-scales that are many orders of magnitudes longer than the thermal relaxation time between matter and radiation – in most stellar layers and most of the time. Exceptions, as always when they occur, tend to be relevant and should be given appropriate attention. In the current context, the exceptions belong to dynamical changes on very short time-scales, such as explosive burnings and/or dynamical phenomena in low-density regions, such as winds, violent pulsations with shocks, or accretion shocks in protostars, just to name a few (for explicit and illuminating examples, consult the book of Mihalas & Mihalas-Weibel 1984).

The EoS physics has to deal with aspects such as condensation, dissociation, ionization, plasma interaction, crystallization and degeneracy.

Opacity: In astrophysics, the atomic data specifying the opacity is usually presented in tabular form so that we can easily obtain opacity

$$\kappa = f(\rho, T, \boldsymbol{\chi}).$$

As we are talking about atomic physics here, opacity computations deal with the quantum mechanics of ideal and real gases (perturbations of population levels by neighboring atoms). To correctly calculate the level populations opacity computations need consistent and accurate EoS treatments.

Nuclear energy generation: The net output of energy produced by subatomic processes nuclear energy generation

$$\varepsilon = f(\rho, T, \boldsymbol{\chi}) \equiv \varepsilon_{\mathrm{nuc}} - \varepsilon_{\nu}$$

adds up all nuclear and particle physics aspects (i.e. strong and weak interactions) contributing to the star's energy budget. In the stellar-structure setting, plasma interactions, e^- screening, degeneracy, and even solid-state effects (pycno-nuclear burning) must be given appropriate attention.

In traditional stellar evolution codes, the spatio-temporal evolution of the nuclear species is separated from the spatial structure computation (numerically this turned out to be very wise, or more probably it was even enforced at the time of writing the first generation of codes!). Formally, we can describe the evolution of a nuclear species X_i with the *diffusion ansatz* species evolution

$$\mathrm{D}_t X_i = \partial_m \left(\sigma_{\mathrm{D}_i} \partial_m X_i\right) + Q_i - S_i. \tag{16}$$

The sink S_i and the source Q_i are determined by nuclear-burning processes. All the microphysics of *species transport* (convective and semi-convective mixing, radiative levitation, sedimentation, ...) is hidden in the diffusion coefficient σ_{D_i}.

The stellar structure and evolution equations define an *initial – boundary-value problem* with two-sided boundary conditions

$$
\begin{aligned}
r &= 0,\\
u &= 0,\\
L &= 0
\end{aligned}
$$

at $m = 0$ and

$$
\begin{aligned}
L &= 4\pi R^2 \sigma T^4,\\
P &= f(\rho, T, \kappa)
\end{aligned}
$$

at $m = M_*$. The first equation at the outer boundary is usually referred to as the Stefan-Boltzmann condition. The second equation says formally that the pressure at the photosphere is determined by a more or less complicated atmosphere approximation; most members of the stellar-evolution curia still prefer the less complicated flavor of applying a simple analytical relation for the second equation. As it is well known from basic stellar-evolution courses ,however: In cool stars with extended convective envelopes, any laziness tolerated at the outer boundary propagates deep into the interior.

Finally, initial conditions, such as $M_* = M$ and $\boldsymbol{\chi}(m)$ need to be specified at some $t = t_0$.

The time-scale hierarchy in the stellar structure equations considerably simplifies dealing with them in the course of numerical simulations:

dynamical time-scale	τ_{ff} :	$\mathcal{O}(n \cdot \mathrm{sec}) - \mathcal{O}(m \cdot \mathrm{months})$
Kelvin-Helmholtz time-scale	τ_{KH} :	$\mathcal{O}(10^7 \mathrm{y})$
nuclear time-scale	τ_{nuc} :	$\mathcal{O}(10^9 \mathrm{y})$

Focus for the moment on the acceleration term in the momentum equation (13):

$$\frac{\text{gravitational term}}{\text{dynamical term}} \propto \frac{Gm}{r^3} t^2 \propto \left(\frac{t}{\tau_{\mathrm{ff}}}^2\right) \tag{17}$$

For the majority of all stellar-evolutionary phases being computed, the time steps that are taken are

$$\Delta t_{\mathrm{evol}} \approx \tau_{\mathrm{KH}} \quad \text{or larger.}$$

Use this Δt_{evol} as the characteristic time in eq.(17). Hence, the approximation

$$\partial_m P = -\frac{Gm}{4\pi r^4} \tag{18}$$

quasi hydrostatics

is a very good one. BE AWARE, however, not to miss the most interesting and rewarding effects in stellar evolution by shutting off time dependences too early. By suppressing the acceleration terms *always*, stellar evolution folks would never have encountered the classical instability strip or core-collapse supernovae, for example. The onset of new nuclear burning phases after phases on the Kelvin-Helmholtz time-scale might look different when computed with acceleration dragged along than when quasi-hydrostatically bulldozing them.

> Marginalia: Note the formal inconsistency in the stellar evolution equations: $\mathbf{F} \neq 0$, but $\Psi \equiv 0$ and $\mathbb{P} \equiv \mathbf{0}$. Stellar evolution is, hence, only partially Navier-Stokesian. But, as it is customary today: Do not feel bad about doing wrong: The effects due to this inconsistency are marginal; mostly because of the small velocities developing in stellar structure context.

Stellar physics and structure texts are rather abundant and some are outstanding. Microphysics of all flavors and relevant for stellar environments is treated comprehensively in Cox and Giuli (1968). Macro-structure analyses of stars are paid little attention to in the literature; this is overcome by the monograph of Kippenhahn and Weigert (1994). For a taste of early attempts to numerically compute stars and enlightening physical argumentations, I suggest to delve into (Schwarzschild, 1958). For heavy semi-analytics, applied math, and rigorous descriptions of early views on degenerate stars, consult Chandrasekhar (1939).

homology invariants

Oldies but Goodies: The homology invariants U and V turn out to be handy in the following. These double logarithmic derivatives characterize the mechanical structure of a star. The invariant U defines the mass stratification

$$U = \frac{\mathrm{d}\log m}{\mathrm{d}\log r} \equiv \frac{4\pi r^3 \rho}{m}\,, \tag{19}$$
$$V = -\frac{\mathrm{d}\log P}{\mathrm{d}\log r} \equiv \frac{\rho G m}{P r}\,. \tag{20}$$

The invariant V describes the pressure stratification in a star and is closely related to the pressure scale height $H_P \doteq -\mathrm{d}r/\mathrm{d}\ln P$.

Conservative perturbation theory: To keep matters simple to begin with, a hypothetical stellar model is perturbed such that neighboring fluid elements do **not** exchange heat during the perturbation process. Such a conservativity constraint on the processes is called *adiabatic* or less frequently *isentropic*.

Any physical quantity q is decomposed as follows

$$q(m,t) = q_0(m) + \Delta q(m)\exp(\mathrm{i}\omega t)\,. \tag{21}$$

The perturbed quantity, Δq, is called the *lagrangian* perturbation of q, it describes the state of the perturbed fluid element which usually is displaced from its original position in the star. In linear pulsation theory, $\Delta q/q \ll 1$ is always implicitly assumed. lagrangian perturbation

The discussion of adiabatic radial perturbations needs the following perturbed quantities

$$\frac{\Delta r}{r} = x\,, \quad \frac{\Delta P}{P} = p\,, \quad \frac{\Delta T}{T} = t\,, \quad \frac{\Delta L}{L} = l\,. \tag{22}$$

Since the processes are assumed to be adiabatic during the perturbation, pressure is directly coupled to the density

$$P \propto \rho^{\Gamma_1} \quad \text{or linearized} \quad h = \frac{1}{\Gamma_1} p\,, \tag{23}$$

using the definition $h = \Delta\rho/\rho$.

Inserting the decomposition (21) into the continuity equation (12), subtracting the equilibrium solution, using the relations in (22) and (23) and neglecting products of perturbed quantities leads to

$$\mathrm{d}_m x = -\frac{1}{4\pi r^3 \rho}\left(3x + \frac{1}{\Gamma_1} p\right)\,. \tag{24}$$

For ease of notation, subscripts 0 were and will be neglected in equilibrium quantities, such as in r, ρ and Γ_1 in eq. (24) whenever no confusion is likely to arise.

The linearization of the momentum equation (13) follows the line of reasoning used in the last paragraph. Additionally, the oscillation frequency ω is preferably expressed in units of the star's free-fall frequency $\sqrt{3GM_*/R_*^3}$, i.e.

$$\sigma^2 \doteq \frac{\omega^2}{(3GM_*/R_*^3)}\,. \tag{25}$$

Furthermore, a measure of the mass concentration is introduced:

$$c_1 \doteq \frac{(r/R_*)^3}{m/M_*}\,.$$

Therewith, the linearized momentum equation can be written as

$$\mathrm{d}_{\ln m} p = \frac{V}{U}\, p + \frac{V}{U}\left[3c_1\sigma^2 + 4\right] x\,, \tag{26}$$

or if radius r serves as the independent variable

$$\mathrm{d}_{\ln r} p = V\, p + V\left[3c_1\sigma^2 + 4\right] x\,. \tag{27}$$

Combining eqs. (24) and (27) leads to a second-order differential equation for adiabatic perturbations. To shorten the notation, the replacement $(.)' \equiv \mathrm{d}_{\ln r}$ will be used furtheron. LAWE

$$x'' + (3 - V)\,x' + \frac{V}{\Gamma_1}\left[3\,c_1\,\sigma^2 + (4 - 3\Gamma_1)\right]\,x = 0\,. \tag{28}$$

The particular form of eq. (28) resulted from the additional assumption of a spatially constant Γ_1. This constraint does not suppress important qualitative features of the solutions, so we stick to it for simplicity. Equation (28) is also known as the Linear Adiabatic Wave Equation (LAWE).

Equation (28) constitutes a *boundary-eigenvalue problem* of second order; for a complete definition of the problem, two boundary conditions must also be supplied: For physical reasons, the solution must be regular in the stellar center, i.e. x must not diverge faster than with $1/r$ as the stellar center is approached. The solution, x, is also required to be regular at $r = R_*$, at the stellar surface. The eigenvalue of the problem is σ^2.

The properties of the stellar models enter the problem via the coefficients c_1, V, and Γ_1. In most cases, these quantities are no analytically known functions but are known on a number of grid points of some numerically calculated stellar model only.

The only case of a self-gravitating sphere that admits a closed-form solution to eq. (28) is the one with constant density. Even if such a configuration is not of much astronomical relevance, it has the benefit of demonstrating important results that carry over to more complex situations in an easily comprehensible way.

Eigenbehavior of radially perturbed homogeneous spheres: According to its very name, the homogeneous sphere has constant density throughout its volume, i.e. $\rho = \text{cst.}$. The hydrostatic equation can be easily integrated, resulting in

the homogeneous sphere

$$P - P_* = \frac{2\pi}{3} G\rho\left(r^2 - R_*^2\right)\,. \tag{29}$$

For simplicity, we set $P_* \equiv 0$ and we substitute $y = r/R_*$. The homology invariant V and the mass concentration parameter become

$$V = 2\,\frac{y^2}{1 - y^2} \quad \text{and} \quad c_1 = \frac{\langle\rho\rangle}{\rho} \equiv 1\,. \tag{30}$$

Finally, the pulsation equation adopts the form

$$\mathrm{d}_y^2 x + \left(\frac{4}{y} - \frac{2\,y}{1 - y^2}\right)\mathrm{d}_y x + \frac{\alpha}{1 - y^2}\,x = 0\,. \tag{31}$$

The new eigenvalue of the problem is α and it relates to σ as follows

$$\alpha \doteq \frac{2\left[3\,\sigma^2 + (4 - 3\Gamma_1)\right]}{\Gamma_1}\,.$$

One solution is evident for $\alpha = 0$: $y = \text{const.}$, i.e. $\Delta r = r_0 \exp(\mathrm{i}\sigma t)$. The nodeless eigenfunction we found so easily is representative of the *fundamental mode* of the problem. Rearranging the definition for α gives the oscillation frequency as

$$\sigma^2 = -\frac{4 - 3\Gamma_1}{3}. \tag{32}$$

Hence, for $\Gamma_1 > 4/3$: $\sigma^2 > 0$, i.e. purely oscillatory solutions prevail. For $\Gamma_1 < 4/3$, $\sigma^2 < 0$ which means that the perturbation Δr diverges exponentially, either in the form of an expansion or contraction. This latter, monotonous instability is called as stellar *dynamical instability* .

dynamical instability

The period of oscillation,Π, relates to the angular frequency like $\Pi = 2\pi/\omega$. The chosen free-fall gauge, cf. eq. (25), leads to

$$\Pi^2 = \frac{\pi}{G\langle\rho\rangle\sigma^2},$$

which leads to the *period – mean-density relation* of pulsating stars. Using the result for the fundamental mode of the homogeneous sphere with its constant amplitude displacement and the eigenfrequency in eq. (32) gives

period - mean-density relation

$$\Pi\,\sqrt{\rho} = \sqrt{\frac{3\pi}{G\,(3\Gamma_1 - 4)}}\,. \tag{33}$$

The right-hand side of eq. (33) is characterized by the stellar material alone via Γ_1, no global or integrated stellar properties appear, at least in the case of the homogeneous sphere. Hence, the pulsation period of the fundamental mode is determined by the ratio of the stellar mass to the radius cubed.

Assuming a constant density sun and setting $\Gamma_1 = 5/3$, we obtain a period of 2.7 h for its fundamental mode. A Cepheid with $\log(M/\mathrm{M}_\odot) = 0.7$ and $\log(R/\mathrm{R}_\odot) = 1.8$ has, on the other hand, a fundamental-mode period of 32 d. For the Cepheid, the assumption of constant density is, however, much worse than even for the sun. So the numbers must be taken *cum grano salis*; nevertheless they show the correct order of magnitude.

Higher radial-overtone modes of the pulsation problem can be obtained via the ansatz:

$$x = 1 + \sum_i b_i y^i .$$

In particular, the first-overtone mode is represented as $x = 1 + by^2$; inserting this decomposition into eq. (31) gives $b = -7/5$ and $\alpha = 14$. Hence,

$$\sigma_1^2 = \frac{10\Gamma_1 - 4}{3} .$$

The ratio of the periods of the first overtone to the fundamental mode is

$$\frac{\sigma_0^2}{\sigma_1^2} = \frac{\Pi_1^2}{\Pi_0^2} = \frac{4 - 3\Gamma_1}{4 - 10\Gamma_1} .$$

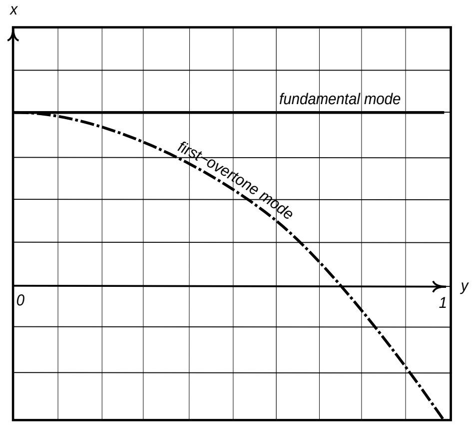

Figure 1: Fundamental and first-overtone radial displacements for a homogeneous sphere.

For $\Gamma_1 = 5/3$, we arrive at $\Pi_1/\Pi_0 = 0.28$. This value of the period ratio is rather far from what is observed in pulsating stars with simultaneously excited fundamental and first-overtone mode: In δ Sct variables, RR Lyraes and Cepheids $\Pi_1/\Pi_0 = 0.7 - 0.75$.

The very magnitude of the period ratio Π_1/Π_0 played an important role in the 1990s when a new generation of *opacity tables* had to be checked for their validity in numerous stellar environments.

Pulsations of polytropic spheres constitute the next step to more realistic stellar models. The price to pay is, however, that analytical results can no longer be obtained.

The derivation of the polytropic stellar-structure equations are not repeated here. If not yet familiar with this simplied approach to compute the mechanical structure of stars, any standard textbook on the theory of stellar structure will remedy this deficit.

In the following, we use the dimensionless dependent variable θ and the independent variable ξ which are related to physical quantities as follows:

$$\rho = \rho_c\, \theta^n, \quad P = P_c\, \theta^{n+1}, \quad \text{and} \quad r = \alpha\, \xi .$$

The quantity n is the polytropic index and

$$\alpha^2 = \frac{K(n+1)\, \rho_c^{\frac{1}{n}-1}}{4\, \pi\, G} .$$

The positive real quantity K is the constant of proportionality between pressure and density, i.e. $P = K\rho^{1+1/n}$.

The *Lane-Emden equation* takes the form: Lane-Emden equation

$$\frac{1}{\xi^2}\,\mathrm{d}_\xi\left(\xi^2 \mathrm{d}_\xi\,\theta\right) = -\theta^n\,. \tag{34}$$

The problem is completely specified with two boundary conditions specified at the center $\xi = 0 : \theta = 1$ and $\mathrm{d}_\xi\theta = 0$. The outer boundary (ξ_1) of the polytropic sphere is reached when θ goes tkrough zero. Analytical solutions are known only for $n = 0, 1, and 5$.

Various types of stars can be modeled relying on polytropic structures, such models are accurate enough, at least, for qualitative studies. A solar-type star, for example, can be approximated with $n = 3$. Giants, with their well pronounced core – envelope structures can be modeled with large n's. As $n \to 5$, the higher the ratio of $\rho_c/\langle\rho\rangle$. Compact stars, such as white dwarfs, are well approximated by $n = 1.5$ polytropes. Hence, the study of pulsating polytropes are useful to gain insight into the canonical behavior of radial eigenfunctions of various classes of stars.

Radial, adiabatic pulsations of polytropes: For polytropic structures, the quantities V and c_1, entering the LAWE, i.e. eq. (28), are written as

polytropes' pulsations

$$V = -(n+1)\,\frac{\xi}{\theta}\,\mathrm{d}_\xi\,\theta, \quad \text{and} \quad c_1 = \frac{\left(\frac{1}{\xi}\mathrm{d}_\xi\,\theta\right)}{\left(\frac{1}{\xi}\mathrm{d}_\xi\,\theta\right)_{\xi_1}}\,.$$

Be

$$\hat{c}_1 \doteq \left(\frac{1}{\xi}\mathrm{d}_\xi\,\theta\right)_{\xi_1}\,.$$

Noting that $\mathrm{d}(.)/\mathrm{d}\ln r = \mathrm{d}(.)/\mathrm{d}\ln\xi$ leads finally to the polytropes' LAWE

$$\mathrm{d}^2_\xi\,x + \left[\frac{4}{\xi} + \frac{n+1}{\theta}\,\mathrm{d}_\xi\,\theta\right]\,\mathrm{d}_\xi\,x + \left[\frac{\varpi^2}{\theta} + \hat{\alpha}\frac{n+1}{\xi\,\theta}\mathrm{d}_\xi\,\theta\right]\,x = 0\,, \tag{35}$$

with $\hat{\alpha} = 3 - 4/\Gamma_1$. The eigenvalue of the problem is now

$$\varpi^2 \equiv -\frac{3\,\hat{c}_1\,\sigma^2}{\Gamma_1}\,.$$

Notice that the coefficients in the two brackets have each simple poles at both boundaries. One boundary condition is applied at each boundary point to select the solutions that are regular at both ends of the interval $[0, \xi_1]$. Since the equation is linear, a third condition is introduced to normalize the solutions to unity (i.e. $x = 1$) at the surface.

Solutions of eq. (35) are, except for asterophysically uninteresting cases, only numerically known. Any numerical boundary – eigenvalue scheme (via finite differences, finite-element Galerkin, or two-sided shooting methods) proved suitable to obtain accurate numerical values.

To demonstrate the particular sensitivities of the eigensolutions selected numerical results are presented in the following. Figure 2 shows the radial displacement amplitudes of the first 5 eigenmodes of an $n = 3$ polytrope with $\Gamma_1 = 5/3$. The displacement amplitudes of all eigenfunctions reach unity at the surface; we cut the plot in fig. 2 to still have sufficiently good resolution of the eigenfunctions in the deep interior of the models.

Counting the nodes of the eigenfunctions shows that a displacement eigenfunction of radial order k has $k-1$ nodes. Compared with the homogeneous sphere, the most conspicuous property of the displacement eigenfunctions for the $n = 3$ polytrope is their low amplitude in the deep interior. The concentration to the surface of the displacements increases with the radial order k. Furthermore, for fixed radial order k, increasing the polytropic index n increases the confinement to the superficial regions of the displacement eigenfunction. For giants and supergiants, even low-order pulsation modes are very superficial phenomena. Hence, for driving and damping processes acting on the oscillation modes, stellar micro-physics close the the surface must be investigated.

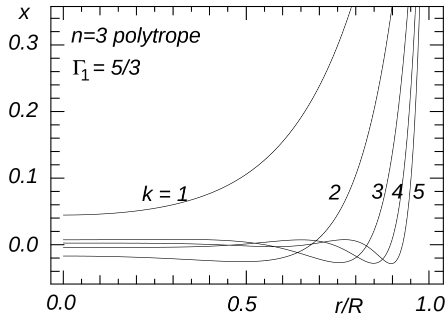

Figure 2: The first five radial displacement eigenfunctions for an $n = 3$ polytrope with an ideal gas having $\Gamma_1 = 5/3$.

Changing the adiabatic index, Γ_1 of the gas affects the fundamental mode most dramatically. Figure 3 shows how the amplitude of the lowest eigenmode increases as Γ_1 is reduced from $5/3$ towards $4/3$, the value of Γ_1 where the squared eigenfrequency of the fundamental mode goes through zero.

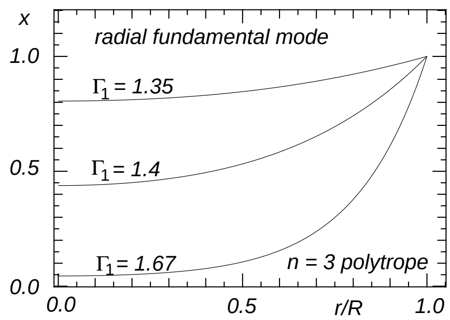

Figure 3: The effect of changing Γ_1 on the radial fundamental mode in a $n = 3$ polytrope.

To finish the section on adiabatic pulsation theory, *generic* properties of the pulsation equations are listed, taking a more mathematical point of view. The propositions are very useful for analytical studies and frequently to check the quality of numerical methods. The proofs of the propositions are left out – a rainy Sunday afternoon should be sufficient, however, for the eager reader to produce them.

Proposition I: All eigenvalues σ^2 are real.

Proposition II: The number of eigenvalues σ_i and eigenfunctions x_i satisfying the LAWE and appropriate boundary conditions is infinite.

Corollary: The eigenvalues are ordered: $\sigma_i < \sigma_{i+1} < \sigma_{i+2}$ and the index i uniquely measures the number of nodes of the eigenfunction.

Proposition III: Choosing an appropriate scalar product, the eigenfunctions x_i, x_j with $i \neq j$ are orthogonal:

$$\int_0^{R_*} \frac{3c_1 r^2}{\Gamma_1} x_i\, x_j \mathrm{d}r' = \delta_{ij}\,.$$

Proposition IV: If $3\,\Gamma_1 - 4 > 0$, then σ_1, the eigenvalue of the fundamental mode, is real; it is imaginary otherwise.

Corollary: If $3\,\Gamma_1 - 4 > 0$, *all* σ_i are real.

Marginalia: Mathematicians and mathematical physicists invested considerable efforts into understanding the properties of the Schrödinger equation in quantum mechanics. The LAWE for stars, on the other hand, went essentially unnoticed for several decades. The missing attention given to the LAWE does not reflect its lack of interesting mathematical properties but rather the unawareness of the appropriate communities.
Beyer (1995) visited the radial adiabatic pulsation problem of polytropes to study it rigorously from the functional analytic viewpoint. In particular, the question if the regular singularities at both endpoints of the interval of interest of the operator could be the origin of a hitherto missed continuous eigenspectrum was answered. Seemingly, the pragmatic physical arguments applied by astronomers over the decades to approximate solutions and picking out the essential spectrum proved correct: no signs of a new continuous spectrum could be found in the rigorous mathematical analysis of the radial LAWE.

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